



Differential Geometry II - Smooth Manifolds

Winter Term 2025/2026

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Exercise Sheet 12

Exercise 1 (*Smoothness criteria for covector fields*):

Let M be a smooth manifold and let $\omega: M \rightarrow T^*M$ be a rough covector field on M . Prove that the following assertions are equivalent:

- (a) ω is smooth.
- (b) In every smooth coordinate chart the component functions of ω are smooth.
- (c) Every point of M is contained in some smooth coordinate chart in which ω has smooth component functions.
- (d) For every smooth vector field X on M , the function $\omega(X): M \rightarrow \mathbb{R}$ is smooth on M .
- (e) For every open subset $U \subseteq M$ and every smooth vector field X on U , the function $\omega(X): U \rightarrow \mathbb{R}$ is smooth on U .

[Hint: Try proving (a) \implies (b) \implies (c) \implies (a) and (c) \implies (d) \implies (e) \implies (b).]

Exercise 2:

Let M be a smooth n -manifold. Show that TM is a smoothly trivial vector bundle if and only if T^*M is a smoothly trivial vector bundle.

Exercise 3 (*Properties of the differential*):

Let M be a smooth manifold and let $f, g \in C^\infty(M)$. Prove the following assertions:

- (a) If $a, b \in \mathbb{R}$, then $d(af + bg) = a df + b dg$.
- (b) $d(fg) = f dg + g df$.
- (c) $d(f/g) = (g df - f dg)/g^2$ on the set where $g \neq 0$.
- (d) If $J \subseteq \mathbb{R}$ is an interval containing the image of f and if $h: J \rightarrow \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f) df$.
- (e) If f is constant, then $df = 0$. Conversely, if $df = 0$, then f is constant on each connected component of M .

Exercise 4:

- (a) *Derivative of a function along a curve:* Let M be a smooth manifold, $\gamma: J \rightarrow M$ be a smooth curve, and $f: M \rightarrow \mathbb{R}$ be a smooth function. Show that the derivative of $f \circ \gamma: J \rightarrow \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

- (b) Let M be a smooth manifold and let $f \in C^\infty(M)$. Show that $p \in M$ is a critical point of f if and only if $df_p = 0$.
- (c) Consider the smooth manifold

$$M := \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$$

and the smooth function

$$f: M \rightarrow \mathbb{R}, (x, y) \mapsto \frac{x}{x^2 + y^2}.$$

Compute the coordinate representation for df and determine the set of all points $p \in M$ at which $df_p = 0$.

Exercise 5 (to be submitted by Thursday, 11.12.2025, 16:00):

- (a) Consider the smooth map

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2, (s, t) \mapsto (st, e^t)$$

and the smooth covector field

$$\omega = x \, dy - y \, dx \in \mathfrak{X}^*(\mathbb{R}^2).$$

Compute $F^*\omega$.

- (b) Consider the function

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}, (x, y, z) \mapsto x^2 + y^2 + z^2$$

and the map

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^3, (u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).¹$$

Compute $F^*(df)$ and $d(f \circ F)$ separately, and verify that they are equal.

- (c) Let M be a compact, connected, smooth manifold of dimension $n > 0$. Show that every exact smooth covector field on M vanishes at least at two points of M .

¹Note that F is the inverse of the stereographic projection from the north pole $N \in \mathbb{S}^2$; see [*Exercise Sheet 2, Exercise 5*].